STUDY OF A LONG RANGE PERTURBATION OF A ONE-DIMENSIONAL KAC MODEL

MARZIO CASSANDRO, IMMACOLATA MEROLA, AND MARIA EULALIA VARES

Abstract. We consider a one dimensional ferromagnetic Ising spin system with interactions that correspond to a $1/r^2$ long range perturbation of the usual Kac model. We apply a coarse graining procedure widely used for higher-dimensional finite range Kac potentials to describe the basic properties of the system and the relation with the mean field theory.

Kac model, long range interaction, Peierls estimates

1. Introduction

We consider a one dimensional Ising spin system on $\mathbb{Z}$ interacting by a long range perturbation of the usual Kac model. More precisely, for a small positive parameter γ, the coupling $J(r)$ between spins at distance r is given by γ if $|r| \leq (2\gamma)^{-1}$, and by λ/r^2 otherwise, where $\lambda > 0$ is fixed. Applying the perturbative scheme around the mean field developed in [6] for finite range Kac potentials in dimensions $d \geq 2$ (see also [15]) and following the notion of contours introduced by Fröhlich and Spencer in [10] as implemented in [5], we study basic properties of this model for small but finite γ.

The main properties of percolation and phase transitions for one-dimensional ferromagnetic models with long range interactions have been established in the seminal papers [10], [14], [2]. Particularly relevant are the results obtained in [10], [1], and [11], where the role of long and short range components of the interactions has been singled out. When $r^2 J(r) \to \lambda \in (0, \infty)$ as $r \to \infty$, they establish the existence of phase transition, prove the discontinuity of the magnetization at the critical point β_c (the so called Thouless effect) and (among other things) determine the limiting value of β_c when a short range interaction, say $J(1)$, tends to infinity. For some related more recent results see e.g. [12, 13].

Our model belongs to this class, except that the strong short range interaction is replaced by the standard finite Kac potential that in the limit $\gamma \to 0$ gives the mean field model with spontaneous magnetization

$$m_\beta = \tanh \beta m_\beta$$

for $\beta > 1$.

The existence of a finite bound for $\beta_c(\gamma, \lambda)$, uniform in γ for γ small follows by putting together the results of [14] for one-dimensional independent site-bond percolation and the inequalities between percolation and Ising models, obtained in [1] through the Fortuin-Kasteleyn representation ([9]). Instead, we will exploit a coarse graining procedure (widely used to study finite range

Kac systems) to get not only a direct proof of this bound, but also a detailed description of the typical configurations. This approach allows to display the relations with the mean field theory. We show the existence of $\bar{\beta}(\lambda)$ so that for all γ sufficiently small:
a) $\beta_c(\gamma,\lambda) < \bar{\beta}(\lambda)$.
b) For all $\beta > \bar{\beta}(\lambda)$, the magnetization under the extremal Gibbs measure with $+1$ (-1) external condition is close to the mean field value m_β $(-m_\beta$ resp.).
c) For all $\beta > \bar{\beta}(\lambda)$ and external conditions cf. Definition 3.1, or still as in b) above, the typical configurations exhibit large intervals (of length $\geq \exp\left(\frac{c(\beta,\lambda)}{\gamma \ln 1/\gamma}\right)$) with magnetization close to $+m_\beta$ or $-m_\beta$ interrupted by fluctuations of the opposite sign of order $o(\gamma^{-1})$.

We believe that with a proper implementation of the multiscale approach introduced by [11], the upper bound $\bar{\beta}(\lambda)$ might be improved and that the spontaneous magnetization should stay close to the mean field value for any $\beta > \beta_c(\gamma,\lambda)$. Nevertheless, it is not clear if the method can be suitably applied to our case. Dealing with coarse grained configurations imposes difficulties in the treatment of Peierls type estimates, and the contour methods implemented here do not provide an optimal result.

2. The model

We consider a spin system on $\mathbb{Z}$: $i \in \mathbb{Z}$ $\sigma(i) \in \mathcal{S}_1 := \{-1,+1\}$. Given $\bar{\sigma} \in \mathcal{S}_1^{\mathbb{Z}}$ and $\Lambda \subset \mathbb{Z}$ finite, the model on $\mathcal{S}_\Lambda := \mathcal{S}_1^\Lambda$ is defined by the Hamiltonian

$$H_\Lambda(\sigma_\Lambda|\bar{\sigma}) = -\frac{1}{2}\sum_{i,j\in\Lambda\cap\mathbb{Z}} J_\gamma(|i-j|)\sigma(i)\sigma(j) - \sum_{\substack{i\in\Lambda\cap\mathbb{Z}\\ j\in\Lambda^c\cap\mathbb{Z}}} J_\gamma(|i-j|)\sigma(i)\bar{\sigma}(j), \quad (2.1)$$

with

$$\begin{aligned} J_\gamma(|i-j|) &:= \gamma \mathbf{1}_{[|i-j|\leq\gamma^{-1}/2]} + \lambda\frac{\mathbf{1}_{[|i-j|>\gamma^{-1}/2]}}{|i-j|^2} \\ &\equiv \gamma(J^{(0)} + \tilde{\lambda}J^{(1)})(\gamma(i-j)), \end{aligned} \quad (2.2)$$

where $\tilde{\lambda} = \lambda\gamma$,

$$J^{(0)}(r) = \mathbf{1}_{[|r|\leq 1/2]} \quad \text{and} \quad J^{(1)}(r) = \frac{1}{r^2}\mathbf{1}_{[|r|>1/2]}, \quad (2.3)$$

and $\mathbf{1}_A$ stands for the indicator function of the set A. The parameter $(2\gamma)^{-1}$, which gives (in microscopic scale) the range of the basic short range mean field interaction, is a positive even integer assumed to be large throughout. The Gibbs measure at inverse temperature β on the finite volume Λ and with external condition $\bar{\sigma}$ is given by

$$\mu_{\Lambda,\beta,\gamma}(\sigma_\Lambda|\bar{\sigma}) = \frac{e^{-\beta H_\Lambda(\sigma_\Lambda|\bar{\sigma})}}{Z_\Lambda(\bar{\sigma})},$$

where

$$Z_{\Lambda,\beta,\gamma}(\bar\sigma) := \sum_{\sigma_\Lambda \in \mathcal{S}_\Lambda} e^{-\beta H_\Lambda(\sigma_\Lambda|\bar\sigma)}.$$

To avoid heavy notation we usually omit the parameter λ that appears in (2.2). Sometimes, whenever no confusion is added, we also omit γ or the inverse temperature β from the notation.

In this paper, we shall always work in the so-called mean-field phase transition region for $J^{(0)}$, i.e., we assume $\beta > 1$ throughout. Let m_β denote the mean field value of the magnetization at temperature $1/\beta$, i.e., the positive solution of the mean field equation:

$$m_\beta = \tanh(\beta m_\beta). \tag{2.4}$$

Given $b > 0$ we write $\tilde\beta(b)$ for the unique value of β that solves

$$\beta m_\beta^2 = b, \tag{2.5}$$

with m_β the non-zero solution of (2.4),

In this context, we have:

Theorem 2.1. *There exists $\bar b$ (independent of λ, γ) so that the following holds:*
a) For any $\lambda > 0$ we can find $\gamma_0(\lambda) > 0$ so that for any $\gamma < \gamma_0(\lambda)$, the system with parameters γ, λ exhibits phase transition and the critical inverse temperature $\beta_c(\gamma,\lambda)$ satisfies

$$\beta_c(\gamma,\lambda) \le \tilde\beta(\bar b/\lambda) =: \bar\beta(\lambda). \tag{2.6}$$

b) If $\beta > \bar\beta(\lambda)$, then

$$\lim_{\gamma\to 0} \mu^+_{\beta,\gamma}(\sigma_0) = m_\beta \tag{2.7}$$

where $\mu^+_{\beta,\gamma} := \lim_{\Lambda\to\mathbb{Z}} \mu_{\Lambda,\beta,\gamma}(\cdot|+\underline{1})$ with $+\underline{1}$ denoting the configurations $\bar\sigma_j = +1$ for all j and, as usual, $\mu(f)$ denotes the integral of f with respect to μ.

Remark. The notion of criticality is the standard one, marking the transition from uniqueness to multiple infinite volume Gibbs measures. The classical Dobrushin uniqueness condition (see [7]) tells that $(1 + 4\lambda\gamma)^{-1} < \beta_c(\gamma,\lambda)$.

Taking into account the result in [1] where it is shown that if $\lim_{r\to\infty} r^2 J(r) = \lambda$ exists, with $0 < \lambda < \infty$, then the following dichotomy holds[1]: $\mu^+_{\beta,\gamma}(\sigma_0) = 0$ or $\mu^+_{\beta,\gamma}(\sigma_0) \ge (2\beta\lambda)^{-1/2}$, we have at once that $\bar b \ge 1/2$. It would be very interesting to extend the analysis to all values of β larger than $\tilde\beta(1/(2\lambda))$, but our techniques do not allow this for the moment. (Our proof works for any $\bar b > 7$.) We should also notice that in [1], see also [2] and [11], the more general context of Potts models is considered.

The plan of the paper is as follows: in section 3 we exploit a coarse graining procedure widely used in the study of Kac systems (see [15]) to describe the

[1]**Notational Remark.** For a given interaction $J(\cdot)$, the Hamiltonian in (2.1) corresponds to twice that in [1, 11].

configurations in terms of $\{-1,0,1\}$-valued spin variables, and state our main theorem in this context. In section 4 we extend to our case the notion of contours introduced in [10], but we follow the implementation given in [5], that is better suited to control the contributions of the zero components of these new spins. In section 5 we prove the upper bound for $\beta_c(\gamma,\lambda)$ via a Peierls argument. In section 6 we prove the free-energy estimates necessary to implement the Peierls argument.

3. Coarse graining

In the sequel we will introduce three new scales, $\ell_0 < \ell_- < \ell_+$, where ℓ_0, ℓ_-/ℓ_0 and ℓ_+/ℓ_- are positive integers, all tending to ∞ as $\gamma \to 0$. We also assume $\ell_0, \ell_-, \ell_+ \in \gamma^{-1}\mathbb{Q}$:

$$\ell_0 := \delta_0 \gamma^{-1} << \ell_- := \delta_- \gamma^{-1} << (2\gamma)^{-1} << \ell_+ := \delta_+ \gamma^{-1}. \tag{3.8}$$

For our proof to work $\delta_0, \delta_-, \delta_+$ should satisfy some relations; an example is given in (6.30).

Notation. For $B \subset \mathbb{Z}$ finite, $|B|$ denotes its cardinality. For any $x \in \ell_*\mathbb{Z}$, where $*$ stands for $0, -$ or $+$, we write $C_x^* = [x, x+\ell_*) \cap \mathbb{Z}$, also called ℓ_*-blocks in the sequel.[2] We also set

$$m^{\ell_*}(x;\sigma) = \frac{1}{|C_x^*|} \sum_{i \in C_x^*} \sigma(i), \qquad \sigma \in \mathcal{S}, \tag{3.9}$$

where now $*$ stands for 0 or $-$. Thus $m^{\ell_*}(x;\sigma)$ takes values in $\{-1, -1+\frac{2}{\ell_*}, \dots, 1\} =: \mathcal{M}_*$. We call $\mathcal{M}_{*,\Lambda} := \mathcal{M}_*^{|\Lambda \cap \ell_* \mathbb{Z}|}$.

For any configuration $\sigma \in \mathcal{S}_\Lambda$ we now define the coarse-grained variables $\eta_\Lambda = (\eta^\psi(x,\sigma) \colon C_x^+ \subset \Lambda)$. The variable $\eta^\psi(x,\sigma)$ provides information on how close the averages $m^-(\cdot,\sigma)$ are to the non-zero solutions of the mean field equation $\pm m_\beta$, over the ℓ_+-block C_x^+. They depend also on a parameter ψ related to the accuracy, and which will be thought as suitably small in comparison with m_β.

$$\eta^\psi(x,\sigma) = \begin{cases} -1, & \text{if} \quad \sup_{y \in C_x^+ \cap \ell_- \mathbb{Z}} |m^{\ell^-}(y;\sigma) + m_\beta| < \psi, \\ \\ +1, & \text{if} \quad \sup_{y \in C_x^+ \cap \ell_- \mathbb{Z}} |m^{\ell^-}(y;\sigma) - m_\beta| < \psi, \\ \\ 0, & \text{otherwise}, \end{cases} \tag{3.10}$$

where we take $\psi = \frac{1}{N}(m_\beta)^2$ for N large.

[2] ℓ_*-block may also refer to any interval that is measurable with respect to the partition generated by the $\{C_x^*\}$, also called ℓ^*-measurable interval.

Remark. The parameter ψ is fixed in the proof below, and therefore sometimes omitted in the notation. On the other hand, a careful examination of the estimates shows that one can indeed make $\psi = \psi(\gamma)$ tend to zero with γ.

Definition 3.1. *We set*

$$\mathcal{S}^{\pm} \quad := \quad \{\sigma : \eta^{\psi}(x;\sigma) = \pm 1, \forall x \in \ell_+\mathbb{Z}\}.$$

Since we will describe the system in terms of the η variables, it is convenient to take Λ as an ℓ_+-measurable interval (i.e. $\Lambda = [h,k) \cap \mathbb{Z}$ for $h, k \in \ell_+\mathbb{Z}$, $h < k$). For notational convenience we also take it centered at 0 in the statement below. Our main theorem is:

Theorem 3.2. *Let $\ell_- = \delta_-/\gamma, \ell_+ = \delta_+/\gamma$ with δ_-, δ_+ chosen as in (6.30). There exists a positive constant $\bar{b}$ such that if $\bar{\beta}(\lambda)$ is defined as the solution of $\lambda\beta m_\beta^2 = \bar{b}$, then for any $\beta > \bar{\beta}(\lambda)$, there exists $\gamma_0 = \gamma_0(\beta,\lambda)$ positive so that for all $\gamma < \gamma_0$ and all $\bar{\sigma} \in \mathcal{S}^+$:*

$$\mu_\Lambda(\eta^{\psi}(0) \neq 1 | \bar{\sigma}) \quad \leq \quad e^{-\beta c'_\beta \psi^3 \ell_-} \tag{3.11}$$

$$\mu_\Lambda(\eta^{\psi}(0) = -1 | \bar{\sigma}) \quad \leq \quad e^{-\beta \tilde{J} \gamma^{-1}}, \tag{3.12}$$

for some $c'_\beta > 0$ and $\tilde{J} > 0$ both depending β ($\tilde{J}$ is the function given in (6.46). The same hold when $\bar{\sigma} = +\underline{1}$, provided $|\Lambda| \geq \exp(2/\gamma)$.

The statements in Theorem 3.2 for $\bar{\sigma} \in \mathcal{S}^+$ are proven at the end of section 4 and the extension to $\bar{\sigma} = +\underline{1}$ is proven at the end of section 6.

Remark. Theorem 2.1 follows at once from Theorem 3.2.

4. Proof of Theorem 3.2

The proof of Theorem 3.2 is obtained by a Peierls contour argument. We first consider the case of $\sigma \in \mathcal{S}^+$, and at the end we discuss how to adapt the estimates to the case when $\bar{\sigma} = +\underline{1}$.

4.1. **Triangles and rectangles.** Given any external configuration $\bar{\sigma} \in \mathcal{S}^+$, we associate to each coarse grained configuration η_Λ in the volume Λ a configuration of "triangles" and "rectangles", whose definition is a natural extension of the one given in [5]. The triangles arise from a geometric procedure (in a plane containing our one-dimensional system) to determine the connection between two interface points, marking a plus or a minus region. (This is also analogous to what happens for usual contours in dimension $d \geq 2$, where the "natural" definition of connection is also appropriate to describe energy fluctuations.)
We start by setting variables $\Theta(h)$ which work as "phase indicators" on the coarse grained lattice. For $h \in \ell_+\mathbb{Z} \setminus \Lambda$, set $\eta(h) = \eta(h,\bar{\sigma}) = +1$ and then

define for $h \in \ell_+\mathbb{Z} \cap \Lambda$

$$\Theta(h) = \begin{cases} -1, & \text{if } \eta(h-\ell_+) = \eta(h) = \eta(h+\ell_+) = -1 \\ +1, & \text{if } \eta(h-\ell_+) = \eta(h) = \eta(h+\ell_+) = +1 \\ 0, & \text{otherwise.} \end{cases}$$

An ℓ_+-measurable interval $[h,k)$ is called *"almost positive"* if

$$\Theta(h) = \Theta(k-\ell_+) = +1 \quad \text{and } \Theta(i) \neq -1 \forall i \in \ell_+\mathbb{Z}, h < i < k-\ell_+.$$

Notice that $\Theta(\cdot) = 0$ is allowed inside an almost positive interval; in particular the magnetization over such an interval might be negative. *Almost negative* intervals are defined analogously.

Definition 4.1 (rectangles)**.** *The rectangles, denoted by the letter Q, are defined as the ℓ_+-measurable intervals that correspond to maximal (non-null) runs of ℓ_+-blocks where $\Theta = 0$.*

Remark. A rectangle of size less than $3\ell_+$ can occur only as a set of two consecutive ℓ_+-blocks with $\eta = -1$ in one block, and $\eta = +1$ in the other one. An isolated h for which $\Theta(h) = 0$ is not possible.

Therefore, a configuration Θ can be regarded as a sequence of maximal "almost positive" and "almost negative" intervals separated by some special rectangles that mark the transition from an interval with a given sign to the next one of opposite sign; such rectangles are then called *interface intervals*. Each Θ-configuration will be represented in terms of "triangles" and "rectangles", the correspondence being bijective once the boundary conditions are fixed. As in [10] and [5], our construction is based on suitably coupling together pairs of interface points. To this end we will use the criterion of minimal distance, which is made geometrically intuitive through a graphical representation where each spin configuration is mapped into a set of triangles and rectangles. The endpoints of the triangles will be pairs of suitable coupled interface points.

The precise location of an interface point is immaterial; for convenience we choose, for each $i \in \mathbb{Z}$, a point $r(i)$ in each interval $[i, i+1/100]$ with the property that for any four distinct integers i_j, $j = 1, .., 4$, $|r(i_1) - r(i_2)| \neq |r(i_3) - r(i_4)|$. This choice is done once for all. If $[h,k)$ is an interface interval, the points r_h and r_k are defined as the corresponding *interface points*, and considered to be paired in the the following construction.

The construction of the triangles is slightly more complicated. We start by attributing colors (blue and red) to each pair of interface points. For an interface interval $[h,k)$: the point r_h is red (blue) if $\Theta(h-\ell_+) = +1$(-1, respectively), in which case r_k will be blue (red, respectively) corresponding to the fact that $\Theta(k) = -1(+1$, respectively).

We let each interface point evolve into a trajectory of the same color, represented in the (r,t) plane by the line $r \pm t$ or $t \geq 0$. The choice between the two directions of the trajectory is made in such a way that each red line (blue line) projects its shadow on the (contiguous) almost positive (almost negative, resp.) interval. We have thus, unless Λ is almost positive (i.e. $\Theta(h) \neq -1$ for all h), a

bunch of growing connected-lines each one emanating from an interface point. Red lines ignore blue lines, and viceversa. When two lines of the same color collide they stop growing and the line which corresponds to the paired interface point (of opposite color) is also canceled. In the meantime, all the other lines keep growing. Our choice of the location of the interface points ensures that collisions occur one at a time so that the above definition is unambiguous.

The process described above will stop in a finite time $t < |\Lambda|$, giving rise to triangles. In fact the collision of two points is represented graphically in the (r,t) plane by a triangle whose basis is the line joining the two interface points and whose sides are the two lines which meet at the time of collision. Triangles will be usually denoted by T.

Definition 4.2 (Triangle)**.** *If $[r_i, r_j]$ is the basis of a triangle in the above construction, the ℓ_+-measurable interval $T = [i,j) \cap \mathbb{Z}$ is called* triangle*.*

Remark. The neighboring external ℓ_+–blocks to the left and to the right of a triangle have equal η-value, $+1$ or -1. This common sign is set as the *sign* of the triangle. Notice that to the left and to the right of a triangle T there are always at least two contiguous ℓ_+-blocks where $\Theta = 0$.

Given any $\bar\sigma \in \mathcal{S}^+$ and $\eta_\Lambda \neq +\underline{1}$, we have defined the Θ configuration and represented it as a collection $(\underline{T}, \underline{Q}) = (T_1, .., T_n, Q_1 \dots, Q_m)$ of triangles and rectangles. When $\eta_\Lambda = +\underline{1}$, we have an empty configuration, hereby denoted by $\emptyset$.

Recalling the definition of $m^{\ell_-}(\sigma)$ in (3.9) we may set:

Definition 4.3 ($\underline{S}(m)$)**.** *For any given boundary condition $\bar\sigma$, and for any $m = m^{\ell_-}(\sigma)$, let $\underline{S}(m;\bar\sigma) = (\underline{T}(m), \underline{Q}(m))$ denote the configuration of triangles and rectangles that correspond to $\eta_\Lambda(m)$.*

Definition 4.4 (distance)**.** *For A and B non-empty subsets of $\mathbb{R}$, $d(A,B)$ denotes the usual distance between the two sets. If Q_1 and Q_2 are rectangles, we set $D(Q_1,Q_2) = d(Q_1,Q_2)$. On the other hand, if T_1 and T_2 are triangles, let $D(T_1,T_2) = d(\mathrm{e}(T_1), \mathrm{e}(T_2))$, where $\mathrm{e}(T)$ denotes the set of extremal points of the interval (in $\mathbb{Z}$) which gives T. Finally, when T is triangle and Q a rectangle we set $D(T,Q) = d(\mathrm{e}(T), Q)$.*

Notice that by construction:

$$D(T_1,T_2) \geq \min\{|T_1|, |T_2|\}, \tag{4.13}$$

$$D(Q_1,Q_2) \geq \ell_+, \tag{4.14}$$

while between a triangle T and a rectangle Q of the same configuration one of the two following relation holds:

$$\begin{aligned} & T \cap Q = \emptyset \ \text{ and } \ D(T,Q) \geq 1 \\ & \text{or} \\ & Q \subset T \ \text{ and } \ D(T,Q) \geq \ell_+. \end{aligned} \tag{4.15}$$

In words, a rectangle can be attached to a triangle only externally.

Definition 4.5 (compatibility)**.** *A given configuration of triangles and rectangles* $(\underline{T}, \underline{Q})$ *(always related to* ℓ_+*–measurable blocks) is called* compatible *if for any couple of elements* [3] *of* $(\underline{T}, \underline{Q})$ (4.13) (4.14), (4.15) *hold, and moreover any rectangle* Q *in this configuration satisfies* $|Q|/\ell_+ \geq 2$*; moreover, when* $|Q| = 2\ell_+$ *then* $Q = [h, h + 2\ell_+)$ *for some* $h \in \ell_+\mathbb{Z}$ *with* $\eta(h)\eta(h + \ell_+) = -1$*.*

In the sequel we denote by $\underline{S} = (\underline{T}, \underline{Q})$ *a set of compatible elements, and by the letter* S *a generic element of such a configuration* $\underline{S}$*. We say that two sets of compatible elements* $\underline{S}_1, \underline{S}_2$ *are compatible (and we use the notation* $\underline{S}_1 \sim \underline{S}_2$*) if* $\underline{S}_1 \cup \underline{S}_2$ *is a compatible configuration.*

Remark (4.13) allows the possibility that (for compatible) $Q_1, T_1, T_2 : T_1 \subset T_2$ or $Q_1 \subset T_2$ (bur not viceversa).

Definition 4.6. *For the external condition* $\bar{\sigma} = +\underline{1}$ *we still define the triangles and rectangles in* Λ *as if the boundary condition were in* $\mathcal{S}_\Lambda^+$*, namely ignoring the interface points on the boundary.*

4.2. **Contours.** As before we first consider the case $\bar{\sigma} \in \mathcal{S}_\Lambda^+$.

Our aim is to apply the Peierls argument to our system. Following [5], for any given configuration $\underline{S} \equiv (\underline{T}, \underline{Q})$ on Λ, we define a partition of $\underline{S}$ by suitably grouping its "elements" (or constituents) S [4] in *contours* $\Gamma^{(i)}$, which should be sufficiently separated from each other, so as to exhibit a weak dependence. This grouping procedure is obtained by an algorithm that creates an hierarchical network of connections, that at the top level identifies groups of connected constituents, namely the contours.

The algorithm $\mathcal{R}(\underline{S})$ on $\{\underline{S}\}$ which associates uniquely to any (compatible) configuration $\underline{S}$ a configuration of contours $\underline{\Gamma} = \{\Gamma_j\}$ is the same defined in [5]. Here we quote only its properties, before which we need the following notation:

We denote by $T(\Gamma)$ the smallest interval which contains all the elements of the contour, its right and left endpoints are denoted by $x_\pm(\Gamma)$. We set:

$$|\Gamma| := \sum_{S \in \Gamma} \frac{|S|}{\ell_+}.$$

Properties of $\mathcal{R}(\underline{S})$

P.0 *Let* $\mathcal{R}(\underline{T}, \underline{Q}) = (\Gamma_1, \ldots, \Gamma_n)$*,* $\Gamma_i = \{T_{q,i} Q_{p,i}, 1 \leq q \leq k_i, 1 \leq p \leq m_i\}$*, then* $\underline{T} = \{T_{q,i}, 1 \leq i \leq n, 1 \leq q \leq k_i\}$*,* $\underline{Q} = \{Q_{p,i}, 1 \leq i \leq n, 1 \leq p \leq m_i\}$*.*

P.1 *Contours are well separated from each other.* All pairs $\Gamma \neq \Gamma'$ verify, for a suitable constant ϖ:

$$D(\Gamma, \Gamma') := \min_{S \in \Gamma, S' \in \Gamma'} D(S, S') > \varpi \ell_+ \min\left\{|\Gamma|^3, |\Gamma'|^3\right\}. \tag{4.16}$$

Condition (4.16) allows $T(\Gamma) \cap T(\Gamma') \neq \emptyset$ in which case either $T(\Gamma) \subset T(\Gamma')$ or $T(\Gamma') \subset T(\Gamma)$; moreover, supposing for instance that the former case is verified,

[3] We abuse language here.

[4] When not needed to distinguish between triangles and rectangles, we use the notation S to denote a generic constituent of the configuration $\underline{S}$. Also we slightly abuse language here.

(in which case we call Γ an inner contour), then for any element $S'_i \in \Gamma'$, either $T(\Gamma) \subset S'_i$ or $T(\Gamma) \cap S'_i = \emptyset$ and $|\Gamma|^3 < |\Gamma'|^3$.
Remark. The constant ϖ has to be taken large enough; in particular we need $\varpi m_\beta^2 > 10$ in the statement of Proposition 4.7.
P.2 *Independence.* Let $\{\underline{S}^{(1)}, \dots, \underline{S}^{(k)}\}$ be k compatible configurations (where $k > 1$) of rectangles and triangles; $\mathcal{R}(\underline{S}^{(i)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i\}$ the contours of the configurations $\underline{S}^{(i)}$. If any two distinct $\Gamma_j^{(i)}$ and $\Gamma_{j'}^{(i')}$ satisfy **P.1**,

$$\mathcal{R}(\underline{S}^{(1)}, \dots, \underline{S}^{(k)}) = \{\Gamma_j^{(i)}, j = 1, \dots, n_i; i = 1, \dots, k\}.$$

It was proved in [5] that not only **P.0**, **P.1** and **P.2** can be actually implemented by some algorithm $\mathcal{R}$, but that such algorithm is unique and therefore there is a bijection between (compatible) configurations of triangles and rectangles and the contours. It is easy to convince oneself that the above mentioned proof holds also for our sets $\{(\underline{T}, \underline{Q})\}$ when rectangles are also present, cf. section 3.1 of [5].

The following two propositions summarize the relevant properties of the contours:

Proposition 4.7. *Let X_Γ denote the event that Γ is a contour in the configuration $\underline{\Gamma}(\sigma)$ on Λ. Then, uniformly in $\Lambda, \bar{\sigma} \in \mathcal{S}_\Lambda^+$, and for γ small enough:*

$$\mu_\Lambda(X_\Gamma|\bar{\sigma}) \le \prod_{T\in\Gamma} (e^{-2\lambda\beta m_\beta^2 a_\beta \ln(|T|\gamma)} e^{-\frac{\beta}{\gamma}(\tilde{J} - 5\tilde{\lambda}\ln 5)}) \prod_{Q\in\Gamma} (e^{-\beta\epsilon\ell - \frac{|Q|}{7\ell_+}} 3^{\frac{|Q|}{\ell_+}}), \tag{4.17}$$

where $a_\beta = (1 - 10/(\varpi m_\beta^2))(1 - \psi/m_\beta)^2$, with ϖ as in (4.16), $\tilde{J}$ and ϵ are given by (6.46) and (6.38) respectively. In particular, if the scales are chosen as in (6.30), then

$$\mu_\Lambda(X_\Gamma|\bar{\sigma}) \le \prod_{S\in\Gamma} e^{-[b_\beta \ln(|S|\gamma) + c(\gamma)]}, \tag{4.18}$$

where $b_\beta = 2\lambda\beta m_\beta^2 a_\beta$ and $c(\gamma) = c(\gamma, \beta) > 0$ tends to ∞ as $\gamma \to 0$.

The proof is given in the next section.

Proposition 4.8. *For all b sufficiently large, $c > \ln 2$, $m \ge 3$:*

$$\sum_{\substack{\Gamma \ni 0 \\ |\Gamma| = m}} \prod_{S\in\Gamma} e^{-b[\ln(|S|\gamma)] - c} \le 2m e^{-b\ln m - (c - \ln 2)} \tag{4.19}$$

Proof. If we do not distinguish between Q and T while counting the contours, the current entropy estimate boils down to Theorem 4.1 in [5], which relies only on the properties **P.0**-**P.2** of the contours. The only difference is an extra combinatorial factor, since each $S \in \Gamma$ can be a Q or a T. This amounts to have an extra factor 2 inside the product over S, which can be easily controlled if $c > ln2$ $\square$

Remark. A careful examination of the proof of Theorem 4.1 in [5] (see also the appendices E and F there) shows that the statement of previous proposition holds for any b such that

$$\sum_m m^6 e^{-\frac{b(1-\rho)}{2}\ln m} < \frac{e^c}{2\varpi}, \tag{4.20}$$

where ϖ is the parameter introduced in (4.16) and ρ is any number arbitrary small. This allows to prove our main theorem for β such that $\lambda\beta m_\beta^2 > 7$.

Proof of Theorem 3.2 for $\bar\sigma \in \mathcal{S}^+$. The definition of contours implies that

$$\mu_\Lambda(\mathbf{1}_{\eta(0)\neq 1}|\bar\sigma) \leq \sum_{\Gamma\ni 0} \mu_\Lambda(X_\Gamma|\bar\sigma). \tag{4.21}$$

From this we see that the proof of (3.11) in Theorem 3.2 follows at once from propositions 4.7, 4.8. The second inequality follows similar lines.

5. Proof of (4.17) of Proposition 4.7

To exploit the mean field limit, we need to express the Gibbs measure in terms of the coarse grained variables $m(\cdot) = m^{\ell_0}(\cdot,\sigma) \in \mathcal{M}_{0,\Lambda}$ introduced in section 3, plus an error term to be controlled if γ is small enough. We write

$$Z_\Lambda(m|\bar\sigma) := \sum_{\sigma_\Lambda:\, m^{\ell_0}(\cdot,\sigma)=m(\cdot)} e^{-\beta H(\sigma|\bar\sigma)} = e^{-\beta\gamma^{-1}[F_\Lambda(m|\bar m)+\mathcal{G}_\Lambda(m|\bar\sigma)]},$$

where $F_\Lambda(m|\bar m)$ and $\mathcal{G}_\Lambda(m|\bar\sigma)$ are defined as follows:

$$\begin{aligned} F_\Lambda(m|\bar m) \;:=\;& \delta_0 \sum_{x\in\Lambda\cap\ell_0\mathbb{Z}} f_\beta(m(x)) \\ &+\frac{(\delta_0)^2}{4\gamma} \sum_{x,y\in\ell_0\mathbb{Z}\cap\Lambda} J_\gamma(|x-y|)(m(x)-m(y))^2 \\ &+\frac{(\delta_0)^2}{2\gamma} \sum_{x\in\ell_0\mathbb{Z}\cap\Lambda, y\in\ell_0\mathbb{Z}\setminus\Lambda} J_\gamma(|x-y|)(m(x)-\bar m(y))^2 \\ &+\frac{(\delta_0)^2}{2\gamma} \sum_{x\in\ell_0\mathbb{Z}\cap\Lambda, y\in\ell_0\mathbb{Z}\setminus\Lambda} J_\gamma(|x-y|)(\bar m(y))^2 \end{aligned} \tag{5.22}$$

with $\bar m(x) := m^{\ell_0}(x;\bar\sigma)$, and f_β the (limiting) mean field free energy

$$\begin{aligned} f_\beta(m) &= -\frac{1}{2}m^2 - \frac{1}{\beta}S(m) \;\text{ with} \\ S(m) &= -\frac{1+m}{2}\ln\frac{1+m}{2} - \frac{1-m}{2}\ln\frac{1-m}{2}, \;\; m\in(-1,1), \end{aligned}$$

$$\mathcal{G}_\Lambda(m \mid \bar\sigma) := \frac{\delta_0}{\beta} \sum_{x\in\Lambda\cap\ell_0\mathbb{Z}} S(m(x)) + \frac{1-\|J_\gamma\|_0}{2}\delta_0 \sum_{x\in\Lambda\cap\ell_0\mathbb{Z}} (m(x))^2 \tag{5.23}$$
$$-\frac{\gamma}{\beta}\ln\Bigg(\sum_{\sigma: m^{\ell_0}(\sigma)=m} \exp\Bigg\{ -\frac{\beta}{2} \sum_{\substack{x,y\in\ell_0\mathbb{Z}\cap\Lambda\\ i\in C^0_x, j\in C^0_y}} (J_\gamma(|i-j|) - J_\gamma(|x-y|))\sigma_i\sigma_j$$
$$-\beta \sum_{\substack{x\in\ell_0\mathbb{Z}\cap\Lambda, y\in\ell_0\mathbb{Z}\setminus\Lambda\\ i\in C^0_x, j\in C^0_y}} (J_\gamma(|i-j|) - J_\gamma(|x-y|))\sigma_i\bar\sigma_j \Bigg\}\Bigg),$$

where

$$\|J_\gamma\|_0 := \ell_0 \sum_{i\in\mathbb{Z}} J_\gamma(i\ell_0).$$

Notice that $\|J_\gamma\|_0 = 1 + O(\delta_0)$ and that $J^{(1)}$ does not contribute to the mean field limit, as seen at once from (2.2). For any $\bar\sigma \in \mathcal{S}^+$ and $\underline{\Gamma}$ a compatible configuration of contours in Λ, we write:

$$\hat H_{\bar\sigma}(\underline{\Gamma}) = -\frac{\gamma}{\beta}\ln \sum_{m\in\mathcal{E}_\Lambda(\underline{\Gamma})} e^{-\beta\gamma^{-1}[F_\Lambda(m|\bar m)+\mathcal{G}_\Lambda(m|\bar\sigma)]}, \tag{5.24}$$

where $\mathcal{E}_\Lambda(\underline{\Gamma})$ stands for set of possible profiles $m^{\ell_0}(\cdot)$ which give rise to such configuration of contours. Hence,

$$Z_\Lambda(\bar\sigma) = \sum_{\underline{\Gamma}} e^{-\beta\gamma^{-1}\hat H_{\bar\sigma}(\underline{\Gamma})} \tag{5.25}$$

where the sum is over all the possible sets of compatible configurations of contours in Λ.

Notation. Since $\bar\sigma \in \mathcal{S}^+$ is fixed in the derivation below and the estimates do not depend on its value, we omit the subscript in $\hat H_{\bar\sigma}(\underline{\Gamma})$ in the sequel.

The next lemmas summarize basic properties of $\hat H(\underline{\Gamma})$. The first will be proven in the next section, and the second follows easily from the same arguments as in [5].

Lemma 5.1. *Let $(\underline{T}, \underline{Q})$ be a configuration in Λ with a unique contour Γ_0. Then, for γ small enough:*

$$\hat H(\Gamma_0) - \hat H(\emptyset) \geq W(\Gamma_0) := \sum_{Q\in\Gamma_0} \frac{\delta_{-}\epsilon}{7}\frac{|Q|}{\ell_+} + \sum_{T\in\Gamma_0}\left(2\tilde\lambda(m_\beta-\psi)^2\ln(|T|\gamma) + \tilde J - 5\tilde\lambda\ln 5\right), \tag{5.26}$$

where $\tilde J$ and ϵ are the same as in Proposition 4.7.

Lemma 5.2. *Let $\underline{\Gamma}\cup\Gamma_0$ be a compatible configuration of contours, then:*

$$\hat H(\underline{\Gamma}\cup\Gamma_0) - \hat H(\underline{\Gamma}) \geq W(\Gamma_0)\left(1 - \frac{10}{\varpi m_\beta^2}\right) \tag{5.27}$$

$W(\Gamma_0)$ *the same as defined in Lemma 5.1 and* ϖ *is the same constant appearing in* (4.16).

Proof. The proof is based on the property **P.1** in the definition of contours, which allows to neglect the interactions between contours, when ϖ is chosen sufficiently large so that ϖm_β^2 is larger than 10. For the explicit estimates we refer to section 3.2 in [5]. □

Proof of proposition 4.7. If we notice that uniformly in $\Lambda, \bar{\sigma} \in \mathcal{S}_\Lambda^+$, Γ_0:

$$\mu_\Lambda(X_{\Gamma_0}|\bar{\sigma}) = \frac{\sum_{\underline{\Gamma}\sim\Gamma_0} e^{-\beta\gamma^{-1}\hat{H}(\underline{\Gamma}\cup\Gamma_0)}}{Z_\Lambda(\bar{\sigma})} \leq e^{-\beta\gamma^{-1}\inf_{\underline{\Gamma}\sim\Gamma_0}[\hat{H}(\underline{\Gamma}\cup\Gamma_0)-\hat{H}(\underline{\Gamma})]},$$

the proof of (4.17) follows from Lemma 5.2 given that the parameter ϖ has been chosen as above. □

Strategy of the proof of Lemma 5.1

We are assuming that there is a unique contour Γ. It consists in a set of triangles and rectangles $\Gamma = \{S_i\}_{i=1,\dots,n}$. We will proceed in the estimate of Lemma 5.1 iteratively "removing" elements one at a time. Namely re-writing the l.h.s of (5.26) as:

$$\begin{aligned} [\hat{H}(\Gamma) - \hat{H}(\emptyset)] &\geq [\hat{H}(\Gamma) - \hat{H}(\emptyset; \mathcal{A}(\Gamma)] \\ &\equiv \sum_{i=1}^{n}[\hat{H}(\Gamma \setminus \cup_{j<i}S_j; \mathcal{A}(\cup_{j<i}S_j)) - \hat{H}(\Gamma \setminus \cup_{j\leq i}S_j; \mathcal{A}(\cup_{j\leq i}S_j))] \\ &\equiv \sum_{i=1}^{n} W_\Gamma(S_i) \end{aligned} \tag{5.28}$$

where, for any subset $\mathcal{D} \subset \mathcal{M}_{0,\Lambda}$ we define:

$$\hat{H}(\underline{\Gamma}; \mathcal{D}) := -\frac{\gamma}{\beta}\ln \sum_{m\in\mathcal{E}_\Lambda(\underline{\Gamma})\cap\mathcal{D}} e^{-\beta\gamma^{-1}[F_\Lambda(m|\bar{m})+\mathcal{G}_\Lambda(m|\bar{\sigma})]}, \tag{5.29}$$

and we will define in the sequel suitable choices for the sets $\mathcal{A}(\cup_{j<i}S_i)$ depending only on the support of $\cup_{j<i}S_i$, and such that $\hat{H}(\emptyset) \leq \hat{H}(\emptyset; \mathcal{A}(\Gamma))$.

Definition 5.3. *A rectangle Q in a configuration $(\underline{T}, \underline{Q})$ is called* "isolated rectangle" *if $D(Q,T) \geq \ell_+$ for any triangle T in $(\underline{T}, \underline{Q})$. Otherwise we say that it is an* "attached rectangle"*.*

In the next section we will prove these basic estimates, namely the lower bound for the two types of elements S, labeled as outlined above.

6. Basic estimates

Let $\Gamma = \{S_i\}_{i=1,\dots n}$ be a contour, which we assume to be the only contour of the whole configuration in Λ.
Choice of the scales. The choice of $\delta_0, \delta_-, \delta_+$ as functions of γ is not strict, but we fix here some suitable values (in terms of γ) as follows:

$$\delta_0 = \gamma^{1/2} \qquad \delta_- = \frac{1}{\ln \gamma^{-1}} \qquad \delta_+ = \gamma^{-1/2}\frac{1}{(\ln \gamma^{-1})^3}. \tag{6.30}$$

Given $\epsilon > 0$, the following inequalities hold for γ sufficiently small:

$$\delta_-\epsilon > \frac{\delta_+}{\ell_0} \tag{6.31}$$

$$\delta_-\epsilon > \tilde{\lambda} \ln \delta_+ \tag{6.32}$$

$$\delta_-\epsilon > \frac{\delta_+}{\ell_0} \ln \ell_0 \tag{6.33}$$

$$\delta_-\epsilon > \delta_0\delta_+. \tag{6.34}$$

In particular, for γ sufficiently small and for any n, $n\delta_-\epsilon > \tilde{\lambda}\ln(n\delta_+)$. (We use ϵ as in (6.38).)
Remark. In the following derivations, $c, c', \dots$ indicate constants whose value is not truly important (even if depending sometimes on the parameter β) and may change from place to place.

. *The first basic estimate*

Following the strategy outlined in (5.28) when S_1 is an isolated rectangle and $\mathcal{A}(S_1) = \mathcal{M}_{0,\Lambda}$, the whole set of possible profiles m. Hence $\hat{H}(\Gamma; \mathcal{M}_{0,\Lambda}) \equiv \hat{H}(\Gamma)$.

Lemma 6.1. *There exist a constant c'_β so that for any isolated rectangle Q and any contour Γ which contains Q:*

$$\hat{H}(\Gamma) - \hat{H}(\Gamma \setminus Q) \geq \frac{|Q|}{6\ell_+} c'_\beta \delta_- \psi^3. \tag{6.35}$$

Remark 1 Let $Q = [h, k)$ with $h, k \in \ell_+\mathbb{Z} \cap \Lambda$ be an isolated rectangle. By definition $\Theta(j) = 0$ for any $j \in Q \cap \ell_+\mathbb{Z}$ and $\Theta(h - \ell_+) = \Theta(k) \neq 0$. Let us assume, without loss of generality that this common value is $+$, the opposite case being completely analogous. In this case the following holds for the η-variables:

$$\eta(h - 2\ell_+) = \eta(h - \ell_+) = \eta(h) = \eta(k - \ell_+) = \eta(k) = \eta(k + \ell_+) = +1$$

and $|\eta(i - \ell_+) + \eta(i) + \eta(i + \ell_+)| \leq 2$ for any $i \in [h, k - \ell_+) \cap \ell_+\mathbb{Z}$, i.e., that there are no three consecutive C^+ blocks where η has the same sign.

Remark 2 In each C^+_h block with $\eta(h) = 0$, we partition the set of possible configurations $m = m^{\ell_0}(\cdot, \sigma)$ as follows:

(a) $\mathcal{A}_h := \{m^{\ell_-} : \eta(h) = 0, \sup_{x \in C^+_h \cap \ell_-\mathbb{Z}} \big(\big||m^{\ell_-}(x, \sigma)| - m_\beta\big|\big) > \psi\}$

(b) $\mathcal{B}_h := \{m^{\ell_-} : \eta(h) = 0, \sup_{x \in C^+_h \cap \ell_-\mathbb{Z}} \big(\big||m^{\ell_-}(x, \sigma)| - m_\beta\big|\big) \leq \psi\}$.

We then denote by

$$\epsilon_a \;:=\; \inf_{m: m^{\ell_-}(m)\in\mathcal{A}_h} F_{C_h^+}(m|m) - F_{C^+}(m_\beta|m_\beta) \tag{6.36}$$

$$\epsilon_b \;:=\; \inf_{m: m^{\ell_-}(m)\in\mathcal{B}_h} F_{C_h^+}(m|m) - F_{C^+}(m_\beta|m_\beta). \tag{6.37}$$

Remark 3 Notice that, since the interaction energy is positive (so that enlarging the size of region does not lower the minimum), the free energy of two contiguous cubes with opposite signs of the variable η is bigger than or equal to ϵ_b

Remark 4 By the previous remarks $1, 2, 3$ we have that the minimal free energy of a rectangle composed by n C^+-blocks is at least $\min\{\epsilon_a, \epsilon_b\}\frac{n}{3}$. In appendix 6.2 the following proposition will be proved:

Proposition 6.2. *Let ϵ_a, ϵ_b be defined as in the Remark 2 above. Then:*

$$\epsilon := \frac{\min\{\epsilon_a, \epsilon_b\}}{\delta_-} > c'_\beta \psi^3 \tag{6.38}$$

where c'_β is a positive constant.

Proof of Lemma 6.1. **Notation.** For $m \in [-1, 1]$, $(m)_\gamma$ denotes the best approximation of m in $\mathcal{M}_0$.

In the previous notation, i.e. writing $Q = [h, k)$ for the isolated rectangle as before, we set (for any configuration $m \in \mathcal{M}_{0,Q}$)

- $$\tilde{m}(x; m) = \begin{cases} m(x), & \text{if } x \notin \mathcal{B}(Q) \\ (\phi_{\mathcal{B}(Q)}(x; m_{[\mathcal{B}(Q)]^c}))_\gamma & \text{if } x \in \mathcal{B}(Q) \end{cases} \tag{6.39}$$

  with $\mathcal{B}(Q) := \{x \in Q\colon d(x, Q^c) < \ell_+\}$ and $\phi_A(x; m_{A^c})$ the function defined in Lemma C.1, that has the following property stated in corollary C.2:

  $$|\phi_\Delta(x) - m_\beta| < e^{-c\frac{1}{3}\delta_+} < 1/\ell_0 \qquad \forall x \in \Delta := \{x \in C_h^+ : d(x, [\mathcal{B}(Q)]^c) > \ell_+/3\} \tag{6.40}$$

  where the second inequality in (6.40) follows at once from the choice of the scales (see (6.30)) for γ small.

- $m^* \equiv m^*_Q(m)$, defined as follows:

  $$m^*_Q(x; m) \equiv m^*(x) := \begin{cases} \tilde{m}(x; m), & \text{if } x \notin \hat{Q} \\ s(m_\beta)_\gamma, & \text{if } x \in \hat{Q} \end{cases} \tag{6.41}$$

  where

  $$\hat{Q} := \{x \in Q\colon d(x, [Q]^c) > \ell_+/3\}$$

and $s \in \{-1,+1\}$ is chosen equal to ± 1 depending on the sign of the neighboring blocks.

$$\hat{H}(\Gamma) - \hat{H}(\Gamma \setminus Q) = -\frac{\gamma}{\beta} \ln \frac{\sum_{m\in\mathcal{E}(\Gamma)} e^{-\beta\gamma^{-1}[F(m|\bar{m})+G(m|\bar{\sigma})]}}{\sum_{m\in\mathcal{E}(\Gamma\setminus Q^o)} e^{-\beta\gamma^{-1}[F(m|\bar{m})+G(m|\bar{\sigma})]}} \tag{6.42}$$

$$\geq \inf_{m\in\mathcal{E}(\Gamma)} \{[F_\Lambda(m|\bar{m}) + G_\Lambda(m|\bar{\sigma})] - [F_\Lambda(m^*|\bar{m}) + G_\Lambda(m^*|\bar{\sigma})] - \frac{\gamma}{\beta}\frac{|Q|}{\ell_0}\ln \ell_0. \tag{6.43}$$

By lemma C.1 and corollary C.2 we have that:

$$F_\Lambda(m|\bar{m}) - F_\Lambda(\tilde{m}(m)|\bar{m}) \geq -2c\tilde{\lambda}\ln(\delta_+) - \frac{c\delta_+}{\ell_0}$$

so that:

$$\hat{H}(\Gamma) - \hat{H}(\Gamma \setminus Q) \geq$$
$$\geq \inf_{m\in\mathcal{E}(\Gamma)} \{[F_\Lambda(\tilde{m}(m)|\bar{m}) + \mathcal{G}_\Lambda(m|\bar{\sigma})] - [F_\Lambda(m^*|\bar{m}) + \mathcal{G}_\Lambda(m^*|\bar{\sigma})]$$
$$-2c\tilde{\lambda}\ln(\delta_+) - \frac{\gamma}{\beta}\frac{|Q|}{\ell_0}\ln\ell_0 - c\frac{\gamma|Q|}{\ell_0}.$$

By remarks 4, Proposition 6.2, equation (6.40) and the estimate on $\mathcal{G}_\Lambda(m|\bar{\sigma}) - \mathcal{G}_\Lambda(m^*|\bar{\sigma})$ proved in appendix A we get:

$$\inf_{m\in\mathcal{E}(\Gamma)} \{[F_\Lambda(\tilde{m}(m)|\bar{m}) + \mathcal{G}_\Lambda(m|\bar{\sigma})] - [F_\Lambda(m^*|\bar{m}) + \mathcal{G}_\Lambda(m^*|\bar{\sigma})] > \tag{6.44}$$
$$> \frac{|Q|}{3\ell_+}c'_\beta\psi^3\delta_- - c\frac{\gamma|Q|}{\ell_0} - \frac{\gamma}{\beta}\frac{|Q|}{\ell_0}\ln\ell_0 - c'\tilde{\lambda}\ln(|Q|\gamma) - c\delta_0\gamma|Q|.$$

Recalling the choice of $\delta_0, \delta_-, \delta_+$ (6.30), for γ sufficiently small we get the result (6.35).

Remark This proof does not exploit the hypothesis that the whole configuration consists of a single contour. It is actually true uniformly over all possible configurations compatible with Q. □

The second basic estimate

Lemma 6.3. *Let us assume that Γ is a configuration without isolated rectangles, with a smallest triangle T and two attached rectangles, Q_l, Q_r to the left and the*

right side of T. For all such configurations the following estimate holds true:

$$
\begin{aligned}
\hat{H}(\Gamma) \quad &- \quad \hat{H}(\Gamma \setminus [T \cup Q_l \cup Q_r]) \geq \left[\tilde{J} + \frac{|Q_l|}{6\ell_+}\delta_- \epsilon + \frac{|Q_r|}{6\ell_+}\delta_- \epsilon \right] \\
&+ \quad 2\tilde{\lambda}(m_\beta - \psi)^2 \ln(|T|\gamma) - c'\tilde{\lambda}\ln(|Q_r||Q_l|\gamma^2) \qquad (6.45) \\
&- \quad \gamma \frac{|Q_l| + |Q_r|}{\ell_0}(\frac{1}{\beta}\ln \ell_0 + c) - 4c\tilde{\lambda}\ln(\delta_+) - 5\tilde{\lambda}\ln 4 - 8\tilde{\lambda}K\delta_-
\end{aligned}
$$

where ϵ is given by (6.38),

$$
\tilde{J} := \lim_{\Lambda \to \mathbb{R}} \inf_{m_\Lambda} F^0_\Lambda(m_\Lambda | -[m_\beta]_{\Lambda^c_-}, +[m_\beta]_{\Lambda^c_+}) - F^0_\Lambda(m_\beta | m_\beta) \qquad (6.46)
$$

with the upper index in F^0_Λ indicating the functional calculated for $\lambda = 0$, Λ^c_- and Λ^c_+ denote respectively the left (right) intervals of Λ^c, and c, c', K are constants.

Proof. As in the previous proof, letting $\Delta = T \cup Q_l \cup Q_r$ we have

$$
\begin{aligned}
&\hat{H}(\Gamma) - \hat{H}(\Gamma \setminus [T \cup Q_l \cup Q_r]) \\
&\quad \geq \inf_{m \in \mathcal{E}(\Gamma)} \{[F_\Delta(\tilde{m}(m)|m \circ \bar{m}) + \mathcal{G}_\Lambda(m|\bar{\sigma})] - [F_\Delta(m^*(m)|m \circ \bar{m}) + \mathcal{G}_\Lambda(m^*(m)|\bar{\sigma})]\} \\
&\quad -\frac{\gamma}{\beta}\frac{|\Delta \setminus T|}{\ell_0}\ln \ell_0 - c\frac{|\Delta \setminus T|\gamma}{\ell_0} - 4c\tilde{\lambda}\ln \delta_+,
\end{aligned}
$$

where $m \circ \bar{m}$ is the configuration which agrees with m in Λ and with $\bar{m} = m^{\ell_0}(\cdot, \bar{\sigma})$ outside Λ, $\tilde{m}(m)$ is as in (6.39) with $\mathcal{B}(Q)$ replaced by $\mathcal{B}(Q_l) \cup \mathcal{B}(Q_r)$ and

$$
m^*_{T_0}(x; m) := \begin{cases} m(x), & \text{if } x \notin \Delta \\ -m(x), & \text{if } x \in T \\ (\phi_{B(Q_u)}(x))_\gamma, & \text{if } x \in B(Q_u),\ u = l, r \\ (sm_\beta)_\gamma, & \text{if } x \in Q_u \setminus B(Q_u),\ u = l, r \end{cases}
$$

with $sm_\beta = \pm m_\beta$ according to the sign of the triangle.

The estimate for the contribution of $\mathcal{G}$ is left to the appendix A. We consider only

$$
\begin{aligned}
\inf_{m \in \mathcal{E}(\Gamma)} \{F_\Delta(\tilde{m}(m)|\bar{m}) - F_\Delta(m^*(m)|\bar{m})\} \quad &\geq \quad \inf_{m \in \mathcal{E}(\Gamma)} \{F_{\Delta \setminus T}(\tilde{m}(m)|\bar{m}) - F_{\Delta \setminus T}(m^*(m)|\bar{m})\} \\
&+ \quad \inf_{m \in \mathcal{E}(\Gamma)} \{F_{T,\Delta^c}(\tilde{m}(m)) - F_{T,\Delta^c}(m^*(m))\},
\end{aligned}
$$

where, for A, B disjoint ℓ_0-measurable intervals and $m \in \mathcal{M}_{0,\mathbb{Z}}$:

$$
F_{A,B}(m) = \delta_0 \sum_{x \in A \cap \ell_0\mathbb{Z}} f_\beta(m(x)) + \frac{(\delta_0)^2}{2\gamma} \sum_{\substack{x \in A \cap \ell_0\mathbb{Z} \\ y \in B \cap \ell_0\mathbb{Z}}} J_\gamma(|x-y|)(m(x) - m(y))^2 \qquad (6.47)
$$

Simple computations give the following estimates:

$$\inf_{m\in\mathcal{E}(\Gamma)}\{F_{\Delta\setminus T}(\tilde m(m)|\bar m)-F_{\Delta\setminus T}(m^*(m)|\bar m)\}\geq$$

$$\max\left\{\tilde J,\frac{|Q_r|}{3\ell_+}\delta_-\epsilon\right\}+\max\left\{\tilde J,\frac{|Q_l|}{3\ell_+}\delta_-\epsilon\right\}-c\tilde\lambda\ln(|Q_r||Q_l|\gamma^2)$$
$$-\frac{(|Q_l|+|Q_r|)\gamma c}{\ell_0}$$

$$\inf_{m\in\mathcal{E}(\Gamma)}\{F_{T,\Delta^c}(\tilde m(m)\circ\bar m)-F_{T,\Delta^c}(m^*(m)\circ\bar m)\} \tag{6.48}$$
$$\geq\tilde\lambda(m_\beta-\psi)^2\ln\frac{|T|^2}{|Q_r||Q_l|}-5\tilde\lambda\ln 4-8\tilde\lambda K\delta_-.$$

The computations are carried out in appendix D. □

Proof of the statements in Theorem 3.2 when $\bar\sigma=+\underline{1}$

Having defined the contours as if $\bar\eta\equiv\eta^\psi(\bar\sigma)=+\underline{1}$, the difference in the basic estimates occurs when the contour reaches the boundary of Λ. Since the external $\bar\sigma=+\underline{1}$ favors the appearance of $\eta(h)=0$ close to the boundary, our previous estimates must be modified for contours Γ such that $d(\Gamma,\Lambda^c)=1$. In this case (and now we make explicit the boundary conditions as subindex of $\hat H$), the estimate in (5.26) has to be modified as follows:

$$\hat H_{+\underline{1}}(\Gamma_0)-\hat H_{+\underline{1}}(\emptyset)\geq W_{+\underline{1}}(\Gamma_0)\geq W(\Gamma_0)-\frac{a(\beta,\lambda)}{\gamma}. \tag{6.49}$$

for a suitable $a(\beta,\lambda)$ which can be taken less than 2. Since Proposition 4.8 still holds, the contribution of the contours that contain the origin and touch the boundary is easily controlled, when $|\Lambda|\geq\exp\left(\frac{2}{\gamma}\right)$. This proves the statement.

7. Final comments.

Of course, by the spin flip symmetry, an analogous statement to Theorem 3.2 holds if $\bar\sigma\in\mathcal{S}^-$ (and $\bar\sigma=-\underline{1}$ respectively). As a consequence, and taking sequential limits $\mu_{\Lambda_n}(\cdot|\bar\sigma)$ with $\bar\sigma\in\mathcal{S}^\pm$, we obtain, for $\beta>\bar\beta(\lambda)$, at least two distinct Gibbs measures. It is also known (see [8]) that in the present context all Gibbs measures are translationally invariant. One wonders if these limits do not depend on the specific choice of $\bar\sigma$ and coincide respectively with $\mu^\pm_{\beta,\gamma}$ of Theorem 2.1. Using e.g. the relativized Dobrushin criteria (see [15], [3]) and cluster expansion techniques one should be able to prove this for β large as in this paper.

The techniques used in this paper can also be applied to slower decaying interactions (e.g. $\frac{\lambda}{r^{2-\alpha}}$ for $\alpha<(\frac{\ln 3}{\ln 2}-1)$, see [5]) and boundary conditions $\bar\sigma\in\mathcal{S}^\pm$. In this case (cf. (2.2), (2.3)) $\tilde\lambda=\lambda\gamma^{1-\alpha}$ and b_β in (4.18) tends to infinity as $\gamma\to 0$ for any $\beta>1$. Therefore the Peierls bound holds for any $\beta>1$ and γ sufficiently small, $\beta_c(\gamma)\to 1$ as $\gamma\to 0$ and the analogue of (2.7) is valid for any

$\beta > 1$. The case with boundary conditions $\bar{\sigma} \equiv +1$ is not contained because inequalities (6.49) are not valid.

Acknowledgements: We thank E. Presutti for suggestions and many discussions on Kac models. The authors thank the following institutions for their warm hospitality which made possible for us to carry out this work: *CBPF* and *IMPA*, in Rio de Janeiro, the Universities *La Sapienza* and *Tor Vergata* in Rome, and the *University of L'Aquila*.
Research partially supported by COFIN, Prin n.20078XYHYS Faperj grants E-26/100.626/2007 and E-26/170.008/2008. I.M.'s work was partially supported by GNFM young researchers project "Statistical mechanics of multicomponent systems". M.E.V. is partially supported by CNPq grant 302796/2002-9.

References

[1] M. Aizenman, J. T. Chayes, L. Chayes and C. M. Newman. Discontinuity of the magnetization in one-dimensional $1/|x-y|^2$ Ising and Potts models. Journ. Stat. Phys. **50**, 1–40 (1988).

[2] M. Aizenman and C. M. Newman. Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ models. Commun. Math. Phys. **107**, 611–647 (1986).

[3] F. Baffioni, T. Kuna, I. Merola, E. Presutti. A relativized Dobrushin uniqueness condition and applications to Pirogov-Sinai models. (preprint mp-arc Nr. 04-107 available at http://www.ma.utexas.edu)

[4] F. Baffioni, T. Kuna, I. Merola, E. Presutti. A liquid vapor phase transition in quantum statistical mechanics. (preprint mp-arc Nr. 04-108 available at http://www.ma.utexas.edu)

[5] M. Cassandro, P. Ferrari, I. Merola, E. Presutti. Geometry of contours and Peierls estimates in $d = 1$ Ising models with long range interactions. Journ. Math. Phys. **46** 053305, 2005.

[6] M. Cassandro, E. Presutti. Phase transitions in Ising systems with long but finite range interactions. Markov Proc. Rel. Fields **2**, 241–262 (1996).

[7] R. I. Dobrushin. Prescribing a system of random variables by conditional distributions. Theory Probab. Appl. **15**, 456–458 (1970).

[8] M. P. Fannes, P. Vanheuverszwijn and A. Verberure. Energy-entropy inequalities for classical lattice systems. Journ. Stat. Phys. **29**, 3, 547–560 (1982).

[9] C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model: I. Introduction and relation to other models. Physica **57**, 536 (1972).

[10] J. Fröhlich and T. Spencer. The phase transition in the one-dimensional Ising model with $1/r^2$ interaction energy. Commun. Math. Phys. **84**, 87–101 (1982).

[11] J. Z. Imbrie and C. M. Newman. An Intermediate Phase with Slow Decay of Correlations in One Dimension $1/\left|x-y\right|^2$ Percolation, Ising and Potts Models. Commum. Math. Phys. **118**, 303–336 (1988).

[12] Domingos H. U. Marchetti. Upper bound on the truncated connectivity in one-dimensional $\beta/\left|x-y\right|^2$ percolation models at $\beta > 1$. Rev. Math. Phys. **7**, 723–742 (1995).

[13] D. H. U. Marchetti, V. Sidoravicius and M. E. Vares. Oriented percolation in one-dimensional $1/|x-y|^2$ percolation models. Journ. Stat. Phys, 139, 941–958 (2010).

[14] C. M. Newman and L. S. Schulman. One dimensional $1/|x-y|^s$ percolation models: the existence of a transition for $s \leq 2$, Commun. Math. Phys. **104**, 547–571 (1986).

[15] E. Presutti. *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*, Theoretical and Mathematical Physics, Springer 2009.

Appendices

A. Estimates for $\mathcal{G}(m|\bar\sigma)$

Recall (5.23), which we write as $\mathcal{G}(m \mid \bar\sigma) = \mathcal{G}^{(1)}_\Lambda(m) + \mathcal{G}^{(2)}_\Lambda(m \mid \bar\sigma)$ with

$$\mathcal{G}^{(2)}_\Lambda(m \mid \bar\sigma) := \frac{\gamma}{\beta}\ln \#\{\sigma_\Lambda \in \mathcal{S}_\Lambda : m^{\ell_0}(\sigma_\Lambda) = m\} \tag{A.50}$$
$$-\frac{\gamma}{\beta}\ln\Bigg(\sum_{\sigma:m^{\ell_0}(\sigma)=m}\exp\Bigg\{-\frac{\beta}{2}\sum_{\substack{x,y\in\ell_0\mathbb{Z}\cap\Lambda\\ i\in C^0_x, j\in C^0_y}}(J_\gamma(|i-j|)-J_\gamma(|x-y|))\sigma_i\sigma_j$$
$$-\beta\sum_{\substack{x\in\ell_0\mathbb{Z}\cap\Lambda, y\ell_0\mathbb{Z}\setminus\Lambda\\ i\in C^0_x, j\in C^0_y}}(J_\gamma(|i-j|)-J_\gamma(|x-y|))\sigma_i\bar\sigma_j\Bigg\}\Bigg).$$

Lemma A.1. *Let $\Delta\subset\Lambda$ be an ℓ_+-measurable interval. Assume that $m\in\mathcal{E}_\Delta(\underline{1})$ and $\bar\sigma\in\mathcal{S}^+$. Taking*

$$m^*(x) = \begin{cases} m(x), & \text{if } x\notin\Delta\cap\ell_0\mathbb{Z}\\ -m(x), & \text{if } x\in\Delta\cap\ell_0\mathbb{Z},\end{cases}$$

we have

$$|\mathcal{G}_\Lambda(m|\bar\sigma) - \mathcal{G}_\Lambda(m^*|\bar\sigma)| < 4\delta_0[\tilde\lambda K + 1] \tag{A.51}$$

for some absolute constant K.

Lemma A.2. *Let $\Delta\subset\Lambda$, $m\in\mathcal{E}_\Delta(0)$, and*

$$m^*(x) = \begin{cases} m(x), & \text{if } x\notin\Delta\cap\ell_0\mathbb{Z}\\ \tilde m(x), & \text{if } x\in\Delta\cap\ell_0\mathbb{Z}\end{cases}$$

where $\tilde m\in\mathcal{M}_{0,\Delta}$ is an arbitrary profile. Then

$$|\mathcal{G}_\Lambda(m|\bar\sigma) - \mathcal{G}_\Lambda(m^*|\bar\sigma)| < 2\delta_0(1+\gamma\lambda K) + \gamma(5\lambda + 2\delta_0)|\Delta| \tag{A.52}$$

for K as in (A.51).

Proof of Lemma A.1. Notice that $\mathcal{G}^{(1)}_\Lambda(m|\bar\sigma) = \mathcal{G}^{(1)}_\Lambda(m^*|\bar\sigma)$. Also, due to the spin flip symmetry, the contribution to $\mathcal{G}^{(2)}_\Lambda(m|\bar\sigma) - \mathcal{G}^{(2)}_\Lambda(m^*|\bar\sigma)$ corresponding to $x,y\in\Lambda\cap\ell_0\mathbb{Z}$ vanishes. On the other hand, for $||i-j|-|x-y|| < \ell_0$ we get:

$$|J_\gamma(i,j) - J_\gamma(x,y)| < \max\{\gamma, 5\lambda\gamma^2\}\mathbf{1}_{[||x-y|-1/(2\gamma)|<2\ell_0]} + \frac{3\lambda\ell_0}{|x-y|^3}\mathbf{1}_{[|x-y|>1/(2\gamma)+2\ell_0]}$$

for any $i\in C^0_x, j\in C^0_y$, $x,y\in\ell_0\mathbb{Z}$. Simple computations show that

$$|\mathcal{G}^{(2)}_\Lambda(m|\bar\sigma) - \mathcal{G}^{(2)}_\Lambda(m^*|\bar\sigma)| < \frac{\gamma}{\beta}\ln\left(\frac{\sum_{\sigma:m^{\ell_0}(\sigma)=m} e^{\beta[2\ell_0+\lambda K\delta_0]}}{\#\{\sigma\in\mathcal{S}_\Lambda : m^{\ell_0}(\sigma)=m\}}\right) < \delta_0[2+\lambda K\gamma]$$

where K is an absolute constant. □

Proof of Lemma A.2. Let

$$\mathcal{H}_A(\sigma) := \frac{1}{2}\sum_{x,y\in A\cap\ell_0\mathbb{Z}}\sum_{\substack{i\in C_x^0\\ j\in C_y^0}}(J_\gamma(|i-j|)-J_\gamma(|x-y|))\sigma_i\sigma_j$$

and

$$\mathcal{H}_{A,B}(\sigma) := \sum_{\substack{x\in A\cap\ell_0\mathbb{Z}\\ y\in B\cap\ell_0\mathbb{Z}}}\sum_{\substack{i\in C_x^0\\ j\in C_y^0}}(J_\gamma(|i-j|)-J_\gamma(|x-y|))\sigma_i\sigma_j.$$

We may write[5]:

$$\begin{aligned}
&|\mathcal{G}^{(2)}_\Lambda(m|\bar\sigma)-\mathcal{G}^{(2)}_\Lambda(m^*|\bar\sigma)|\\
&\quad\le\left|\frac{\gamma}{\beta}\ln\left(2\sup_\sigma e^{-\beta\mathcal{H}_{\Delta,\Delta^c}(\sigma\circ\bar\sigma)}\frac{\sum_{\sigma:m^{\ell_0}(\sigma)=m}e^{-\beta(\mathcal{H}_\Delta(\sigma)+\mathcal{H}_{\Delta^c}(\sigma\circ\bar\sigma))}}{\sum_{\sigma:m^{\ell_0}(\sigma)=m^*}\mathrm{e}^{-\beta(\mathcal{H}_\Delta(\sigma)+\mathcal{H}_{\Delta^c}(\sigma\circ\bar\sigma))}}\right)\right.\\
&\qquad\qquad\left.+\frac{\gamma}{\beta}\ln\left(\frac{\sum_{\sigma_\Delta:m^{\ell_0}(\sigma_\Delta)=m^*_\Delta}1}{\sum_{\sigma_{\sigma_\Delta:m^{\ell_0}(\sigma_\Delta)=m_\Delta}}1}\right)\right|\\
&\quad\le\left|\frac{\gamma}{\beta}\ln\left(2\sup_\sigma e^{-\beta\mathcal{H}_{\Delta,\Delta^c}(\sigma\circ\bar\sigma)}\sup_{\sigma_\Delta:m^{\ell_0}(\sigma_\Delta)=m_\Delta}e^{-\beta\mathcal{H}_\Delta(\sigma)}\sup_{\sigma_\Delta:m^{\ell_0}(\sigma_\Delta)=m^*_\Delta}e^{+\beta\mathcal{H}_\Delta(\sigma)}\right)\right|\\
&\quad\le 2\delta_0\Big|(1+\gamma\lambda K)+\gamma|\Delta|\Big|.
\end{aligned}$$

Trivial computations and Stirling formula give

$$|\mathcal{G}^{(1)}_\Lambda(m|\bar\sigma)-\mathcal{G}^{(1)}_\Lambda(m^*|\bar\sigma)|<5\lambda\gamma|\Delta|.$$

□

B. Proof of Proposition 6.2

Proof of the proposition 6.2. Recalling the definitions (6.36), (6.37), we will prove that:

$$\epsilon_a > c'_\beta\psi^3\delta_- \tag{B.53}$$

$$\epsilon_b > \frac{1}{4}m_\beta^2\delta_- \tag{B.54}$$

where c'_β is a positive constant. Once this is proven, and since ψ is chosen as indicated right after (3.10), we get (6.38).

We first prove (B.54). Hence we are assuming that $\eta(h)=0$ and:

$$||m^{\ell_-}(z)|-m_\beta|<\psi\quad\forall z\in C_h^+\cap\ell_-\mathbb{Z}.$$

[5] $\sigma\circ\bar\sigma$ denotes the configuration that agrees with σ in Λ and with $\bar\sigma$ outside Λ.

Recalling (5.22) and writing $m(x)$ for $m^{\ell_0}(x)$, we have:

$$\begin{aligned} F_{C_h^+}(m|\bar m) - F_{C_h^+}(m_\beta|m_\beta) &\ge \frac{\delta_0^2}{2} \sum_{x,y\in C_h^+\cap \ell_0\mathbb{Z}} (m(x)-m(y))^2 \mathbf{1}_{[|\delta_0 x - \delta_0 y|<1/2]} \\ &\ge \sum_{\substack{u,v\in \ell_0\mathbb{Z}\cap C_h^+ \\ |u-v|<\gamma^{-1}/3}} I_{u,v}(m) \end{aligned}$$

where

$$I_{u,v}(m) := \frac{\delta_0^2}{2} \sum_{x\in C_u^-} \sum_{y\in C_v^-} (m(x)-m(y))^2. \tag{B.55}$$

For each u, v such that $|m^{\ell_-}(u) - m_\beta| < \psi$ and $|m^{\ell_-}(v) + m_\beta| < \psi$:

$$\begin{aligned} I_{uv}(m) &\ge \frac{\delta_0^2}{2} \sum_{x\in C_u^-} \sum_{y\in C_v^-} (m(x) - 2m_\beta + 2m_\beta - m(y))^2 \\ &\ge \frac{\delta_0^2}{2} \sum_{x\in C_h^-} \sum_{y\in C_k^-} [(m(x)-m_\beta)^2 + (2m_\beta)^2 + (m(y)+m_\beta)^2 - 6\psi] \\ &\ge \frac{\delta_-^2}{2}[(2m_\beta)^2 - 6\psi]. \end{aligned}$$

Since $\eta(h) = 0$ there are at least $\frac{1}{3\delta_-} - 1$ such pairs (u,v), and we have the following lower bound:

$$F_{C_h^+}(m|\bar m) - F_{C_h^+}(m_\beta|m_\beta) \ge \frac{\delta_-}{4}[(2m_\beta)^2 - 6\psi] \ge \frac{\delta_-}{4} m_\beta^2$$

for $\psi < m_\beta^2/2$.

Let us now consider ϵ_a. In this case, we have at least a block C_z^- where $||m^{\ell_-}(z)| - m_\beta| > \psi$. The main contribution to the free energy in this case comes from the local contribution on the blocks C_x^0 with $x \in \ell_0\mathbb{Z} \cap C_z^-$ where $|m(x) - m_\beta| > \psi$, and/or from the interaction between two blocks C_x^0, C_y^0, $x, y \in \ell_0\mathbb{Z} \cap C_z^-$, with magnetization of opposite signs, close to $\pm m_\beta$. Recalling that $\delta_- < 1/2$, this last term is due only to the short range interaction, which is constant inside each block C^-. We consider a lower bound of the free energy by neglecting all other (non-negative) contributions. For this we set:

$$\begin{aligned} \mathcal{N}_0 &:= \{x \in C_z^- \cap \ell_0\mathbb{Z} \colon ||m(x)| - m_\beta| > \psi/2\} \\ \mathcal{N}_\pm &:= \{x \in C_z^- \cap \ell_0\mathbb{Z} \colon |m(x) - \pm m_\beta| < \psi/2\}. \end{aligned}$$

Let $N_0, N_\pm$ denote the cardinality of the sets $\mathcal{N}_0, \mathcal{N}_\pm$ respectively, and $n_\pm := N_\pm \frac{\ell_0}{\ell_-}$. Hence $N_0 + N_+ + N_- = \frac{\ell_-}{\ell_0}$. It is trivial to verify that due to our condition

on $m^{\ell-}(z)$ we have

$$n_{\pm} \leq \left(1 - \frac{\psi}{4}\right), \tag{B.56}$$

and we can write (with $n_0 = 1 - n_- - n_+$)

$$\begin{aligned} F_{C_z^-}(m|\bar{m}) - F_{C_z^-}(m_\beta|m_\beta) &\geq n_0\delta_-(f(m_\beta+\psi) - f(m_\beta)) + \delta_-^2 n_- n_+(2m_\beta - \psi)^2 \\ &\geq \delta_-(1 - n_- - n_+)c_\beta\frac{\psi^2}{4} + \delta_-^2 n_- n_+(2m_\beta - \psi)^2 \end{aligned}$$

where c_β is a lower bound of the second derivative of the mean field free energy f.

We then can take the minimum of the r.h.s of the above equation, on the set $\{(n_-, n_+) \colon 0 \leq n_\pm \leq \left(1 - \frac{\psi}{4}\right), n_- + n_+ \leq 1\}$, which gives:

$$F_{C_z^-}(m|\bar{m}) - F_{C_z^-}(m_\beta|m_\beta) \;\geq\; \delta_- c'_\beta\psi^3$$

for a suitable positive constant c'_β. In fact, the function $g(x,y) = A(1-x-y) + Bxy$ with $A = \delta_- c_\beta\psi^2$ and $B = \delta_-^2(2m_\beta - \psi)^2$ has a unique critical point in $x = y = \frac{A}{B}$, that, since $B \ll A$, is out of the domain $X + Y \leq 1$. This point is a *saddle*. Evaluating the function on the border of the domain, we get the result: $g(x,y) \geq g(0,(1-\psi/4)) = g((1-\psi/4),0) = A\frac{\psi}{4} = \delta_- c'_\beta\psi^3$. □

C. Proof of equation (6.40)

Lemma C.1. *Let $\Delta \subset \Lambda$ be an ℓ_0-measurable interval. Then there is a constant c and for any $\bar{m}_{\Delta^c} \in \mathcal{M}_{0,\Delta^c}$ there exists an ℓ_0-measurable function $\phi_\Delta(x) \equiv \phi_\Delta(x; m_{\Delta^c})$ so that:*

$$\inf_{m_\Delta \in \mathcal{E}_\Delta(\eta_\Delta = +1)} F_\Delta(m_\Delta|\bar{m}_{\Delta^c}) \geq F_\Delta(\phi_\Delta(x)|\bar{m}_{\Delta^c}) - c\lambda\gamma\ln|\Delta| \tag{C.57}$$

where in (C.57) we $F_\Delta(m_\Delta|\bar{m}_{\Delta^c})$ has been extended to $(-1,1)$–valued profiles in the obvious way.

$$|\phi_\Delta(x; m_{\Delta^c}) - m_\beta| < C\, e^{-c(\beta)\gamma d(x,\Delta^c)} \tag{C.58}$$

uniformly in $\bar{m}_{\Delta^c}$.

Proof of Lemma C.1. After remarking (a) and (b) below, the proof is essentially that in [15] except for the fact that our $J^{(0)}$ is not smooth. The technical details to adapt the proof to this case have already been taken care in [4], see Appendix D there.

(a) For any $\bar{m}_{\Delta^c}(x)$, and if $F^0_\Delta(m_\Delta|\bar{m}_{\Delta^c})$ denotes the functional F_Λ calculated for $\lambda = 0$, one has

$$F_\Delta(m_\Delta|\bar{m}_{\Delta^c}) \geq F^0_\Delta(m_\Delta|\bar{m}_{\Delta^c})$$

(b) $|F_\Delta(m_\Delta|\bar{m}_{\Delta^c}) - F^0_\Delta(m_\Delta|\bar{m}_{\Delta^c})| < c\lambda\gamma\ln|\Delta|$. □

We now state the following corollary of Lemma C.1.

Corollary C.2. *Let us assume that* $\eta(h) = +1$ *and let* $\Delta = C_h^+$. *Then the infimum of* (C.57) *is achieved on a function* ϕ_Δ *that satisfies*

$$|\phi_\Delta(x) - m_\beta| < e^{-c\delta_+/3} =: \epsilon(\gamma) \equiv \epsilon \qquad \text{if } \mathrm{dist}(x, \Delta^c) > \ell_+/3. \tag{C.59}$$

D. Proof of (6.48)

Recall that T is the smallest triangle in the configuration $\underline{\Gamma}(\sigma) = \{\Gamma\}$.

Proof. It follows from equation (6.47) and the symmetry of $f_\beta(m)$ that:

$$F_{T,\Delta^c}(m \circ \bar{m}) - F_{T,\Delta^c}(m^* \circ \bar{m}) \quad = \quad -\gamma(\ell_0)^2 \sum_{\substack{x \in \ell_0\mathbb{Z} \cap T \\ y \in \ell_0\mathbb{Z} \setminus \Delta}} J_\gamma(|x-y|) m(x) m(y),$$

and by Lemma D.1 below we can write:

$$F_{T,\Delta^c}(m \circ \bar{m}) - F_{T,\Delta^c}(m^* \circ \bar{m}) \geq -\gamma(\ell_-)^2 \sum_{\substack{u \in \ell_-\mathbb{Z} \cap T \\ v \in \ell_-\mathbb{Z} \setminus \Delta}} J_\gamma(|u-v|) m^{\ell_-}(u) m^{\ell_-}(v) - 8\tilde{\lambda} K \delta_-.$$

Under the hypothesis of Lemma 6.3, $|m^{\ell_-}(u) + s \cdot m_\beta| < \psi$ for $u \in T$, where $s = \mathrm{sign}(T)$. On the other hand,

$$m^{\ell_-}(v) \begin{cases} \in (s \cdot m_\beta - \psi, s \cdot m_\beta + \psi) & \text{if } v \in I(T) \\ \in \mathcal{M}_- & \text{if } v \notin I(T) \end{cases}$$

from which (6.48) follows easily. ∎

Lemma D.1. *There exist a constant* K *so that for every* A, B *disjoint* ℓ_+*-measurable intervals on* $\mathbb{Z}$ *and every* $m(\cdot) \in \mathcal{M}_{0,\mathbb{Z}}$*:*

$$\left| (\ell_0)^2 \sum_{\substack{x \in A \cap \ell_0\mathbb{Z} \\ y \in B \cap \ell_0\mathbb{Z}}} J_\gamma(|x-y|) m(x) m(y) - (\ell_-)^2 \sum_{\substack{u \in A \cap \ell_-\mathbb{Z} \\ v \in B \cap \ell_-\mathbb{Z}}} J_\gamma(|u-v|) m^{\ell_-}(u) m^{\ell_-}(v) \right|$$
$$\leq \ell_-[4\delta_- \mathbf{1}_{[d(A,B)=1]} + 8\lambda K] \tag{D.60}$$

Proof. The l.h.s. of (D.60) can be written as

$$\left| (\ell_0)^2 \sum_{\substack{u \in \ell_-\mathbb{Z} \cap A \\ v \in \ell_-\mathbb{Z} \cap B}} \sum_{\substack{x \in \ell_0\mathbb{Z} \cap C_u^- \\ y \in \ell_0\mathbb{Z} \cap C_v^-}} \Delta J_\gamma(x,y) m(x) m(y) \right|$$

where for $x \in C_u^-, y \in C_v^-$, $J_\gamma(|x-y|) - J_\gamma(|u-v|)$. Direct calculation shows that:

$$|\Delta J_\gamma(x,y)| \leq \gamma \mathbf{1}_{[||u-v|-1/(2\gamma)| \leq 2\ell_-]} + \frac{8\lambda\ell_- \mathbf{1}_{[|u-v|>1/(2\gamma)-2\ell_-]}}{|u-v|^3}$$

from where (D.60) follows. ∎

Università di Roma "La Sapienza", P.le A. Moro, 00185 Roma, Italy.
E-mail address: marzio.cassandro@roma1.infn.it

Dipartimento di Matematica Pura ed Applicata, Università di L'Aquila, 67100 L'Aquila, Italy.
E-mail address: immacolata.merola@dm.univaq.it

Centro Brasileiro de Pesquisas Físicas (CBPF), Rua Dr. Xavier Sigaud 150, 22290-180 Rio de Janeiro, RJ, Brasil.
E-mail address: eulalia@cbpf.br